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Dynamics on $SL(2, \mathbb{R})\tilde{\otimes}U(1)$

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Abstract. A complete analysis of the quantum dynamical system defined by the quantum group $SL(2, \mathbb{R})\tilde{\otimes}U(1)$ is made. This quantum group, which is primarily related to the relativistic harmonic oscillator, is also shown to provide the quantum dynamics of a free particle moving in $(1+1)$-dimensional anti-de Sitter spacetime. The latter interpretation illustrates the capabilities of the present group approach to quantisation in dealing with dynamical systems in more general spacetimes.

1. Introduction

The use of a group as a basic structure for quantisation provides, as a rule, a precise characterisation of dynamical systems, avoiding certain ambiguities which are present in the more traditional geometric quantisation approaches. Recently, several geometric quantisation techniques have been developed in this spirit (see, e.g., [1, 2]). In a series of papers [3, 4] (for a review, see [5]) we have introduced a group approach to quantisation (GAQ) which is based on associating a dynamical system to a group, the *quantum group*, which has a principal fibred structure. In this paper we carry out the case of the quantisation of a group built on $SL(2, \mathbb{R})$. It constitutes an example in which, only with limited effort, an exact solution is obtained. On the other hand, and as is discussed at the end of the paper, the analysis presented here gives rise to the possibility of studying a wide range of physical systems.

The group of 2×2 unimodular matrices on a field F, $SL(2, F)$ plays a privileged role both in mathmatics and physics. Its Lie algebra is the starting point in the study and classification of the semisimple Lie algebras (see, e.g., [6]) on F, and for $F=\mathbb{C}$ it corresponds to an essential symmetry in physics. Recently, the group $SL(2, \mathbb{R})$ [7] has acquired a special importance in physics because it constitutes (see [8] and

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references therein) the only non-one-dimensional proper subgroup of the Virasoro group, which is essentially the conformal symmetry group in $1+1$ dimensions. The structure of this $SL(2,\mathbb{R})$ subgroup plays a significant role in the classification of the possible representations of the Virasoro group [9, 10]. In $3+1$ spacetime dimensions, on the other hand, $SL(2,\mathbb{R})$ is the subgroup of the 'times' of the conformal group $SO(2,4)$; there exist three different kinematics, each one associated with one of the one-dimensional subgroups of $SL(2,\mathbb{R})$ [11-14]. These ideas have been applied to the study of the dynamical symmetry of the magnetic monopole [15] and have also been combined with supersymmetry [16]. Finally, the group $SL(2,\mathbb{R})$, as the covering group of $SO(1,2)$, may be regarded as the de Sitter or anti-de Sitter group in a two-dimensional $(1+1)$ spacetime.

We devote this paper to discuss some novel aspects of $SL(2,\mathbb{R})$, very closely related to its relevance in de Sitter geometry. We shall consider a pseudoextension (to be defined in section 2) of $SL(2,\mathbb{R})$, $SL(2,\mathbb{R})\tilde{\otimes}U(1)$, as the quantum group corresponding to a relativistic harmonic oscillator in a Bargmann-Fock-Segal-like space. After a suitable change of variables we will also recognise the (quantum) dynamics of a free particle in an anti-de Sitter spacetime (see, e.g., [17]). Although here we restrict ourselves to the two-dimensional anti-de Sitter space the procedure could be repeated for the more physical four-dimensional case. We shall also discuss carefully the different limits of the dynamical system (Inönü-Wigner group contractions) relevant to its structure: the $c\to\infty$ limit, leading to the non-relativistic harmonic oscillator (or particle in non-relativistic anti-de Sitter spacetime) and the $\omega\to 0$ limit (zero frequency, flat metric) leading to the free relativistic particle.

The paper is organised as follows. Section 2 is devoted to defining the quantum group $SL(2,\mathbb{R})\tilde{\otimes}U(1)$ as a pseudoextension, and to a few comments on the GAQ. Section 3 discusses this quantum group as the dynamical group of the quantum relativistic oscillator in a Bargmann-Fock-Segal-like representation. In section 4 we define from the group the configuration space and the anti-de Sitter metric with which it is naturally endowed. In both sections 3 and 4 we compare some apsects of the GAQ with their analogues in the older geometric quantisation scheme, and show how some ambiguities of the latter are solved. The non-relativistic and flat limits are also performed in detail in all the relevant expressions. Finally, in section 5 we comment on a number of quantum groups that could be analysed in connection with gravity.

2. The quantum group $SL(2,\mathbb{R})\tilde{\otimes}U(1)$

The GAQ [3-5] associates a dynamical system with a $U(1)$ principal bundle structure, defined on a group, the quantum group $\tilde{G}$. For systems with a classical limit (see in contrast [18]), the $U(1)$ fibred structure is determined by the fact that $\tilde{G}$ is a central extension of a certain group G. In this paper, $G=SL(2,\mathbb{R})$. This group is simple and the Whitehead lemma [19] establishes that the only extension by $U(1)$ is the direct product one, $G\otimes U(1)$. Nevertheless, it is possible to give the $G\otimes U(1)$ group law in a way which simulates that of a true (non-trivial) central extension. We call a *pseudoextension* of G by $U(1)$, and denote it by $G\tilde{\otimes}U(1)$, to the direct product of G by $U(1)$ when its group law is given in that manner. More precisely, pseudoextensions are defined by 2-coboundaries on G which, in a certain contraction process, become non-trivial 2-cocycles of the contracted group G_c (see [20] for details and the geometrical characterisation of this pseudocohomology). These cocycles define the non-trivial central extensions $\tilde{G}_c$ of G_c by U(1) which constitute the contraction limit of the pseudoextensions $G\tilde{\otimes}U(1)$.

For the case of $G = SL(2,\mathbb{R})$, we can define three different contractions with respect to its three one-parameter subgroups H, each of which is associated with a principal fibration $SL(2,\mathbb{R}) \xrightarrow{\pi} SL(2,\mathbb{R})/H$ of $SL(2,\mathbb{R})$ itself. They define three families of pseudoextensions whose 2-coboundaries ξ_{cob} are locally generated by linear functions $\delta(g)$ on G. These 2-coboundaries may be also globally defined as the pull-back $\xi_{\text{cob}} = \pi^*\gamma$ of the Čech cocycle γ characterising the fibration $SL(2,\mathbb{R}) \xrightarrow{\pi} SL(2,\mathbb{R})/H$ (see [21] for the SU(2) related case). Here we shall restrict ourselves to the case where H is the only compact $SL(2,\mathbb{R})$ subgroup, U(1). We shall write this pseudoextension $SL(2,\mathbb{R})\tilde{\otimes}U(1)$, without making explicit the $SL(2,\mathbb{R})$ subgroup (here U(1), in the general case H) involved in its definition. In the present case there should be no confusion between the U(1) subgroup of $SL(2,\mathbb{R})$ and the U(1) group by which $SL(2,\mathbb{R})$ is pseudoextended.

To find the group law of $SL(2,\mathbb{R})\tilde{\otimes}U(1)$, we require an U(1)-valued 2-coboundary $\xi_{\text{cob}}(g', g)$ on $SL(2,\mathbb{R})$ involving a parameter a which in the contraction limit ($a \to \infty$) becomes a 2-cocycle for the group obtained by contraction from $SL(2,\mathbb{R})$. We shall give the group law of $SL(2,\mathbb{R})$ by means of a chart adapted to the above fibration. Using the fact that $SL(2,\mathbb{R}) \sim SU(1,1)$, the elements $g \in SL(2,\mathbb{R})$ may be characterised by *unimodular* matrices of the form

$$g = \begin{pmatrix} z_1 & z_2^* \\ z_2 & z_1^* \end{pmatrix} \tag{2.1}$$

or by vectors in $\mathbb{C}^2$, $\xi = (z_1, z_2) / \xi^+\sigma_3\xi \equiv \bar{\xi}\xi = 1$. The $SL(2,\mathbb{R})$ group law $g'' = g' * g$ may be obtained either from (2.1) or from $\xi'' = g'\xi$, with the result

$$z_1'' = z_1' z_1 + z_2'^* z_2 \qquad z_2'' = z_2' z_1 + z_1'^* z_2 \tag{2.2}$$

although, because of the unimodularity condition $z_1^* z_1 - z_2^* z_2 = 1$, (2.2) is not given in terms of independent group parameters. The projection π, defined by

$$\pi(\xi) = \pi(z_1, z_2) = (\bar{\xi}\mathrm{i}\sigma_1\xi, \bar{\xi}\mathrm{i}\sigma_2\xi, \bar{\xi}\sigma_3\xi) \equiv (y_1, y_2, y_3) \tag{2.3}$$

maps $SL(2,\mathbb{R})$ onto the two-dimensional positive hyperboloid:

$$\Omega^+ = \{y \in \mathbb{R}^3 / y_3^2 - y_2^2 - y_1^2 = 1, y_3 \geq 1\}. \tag{2.4}$$

Ω^+ is pointwise left invariant by the action of the U(1) subgroup of $SL(2,\mathbb{R})$ (the 'compact' time, see below, defined by the matrices (2.1) with $z_1 = \eta = \mathrm{e}^{\mathrm{i}\beta}$, $z_2 = 0$, under whose action $(z_1, z_2) \to (z_1\,\mathrm{e}^{\mathrm{i}\beta}, z_2\,\mathrm{e}^{\mathrm{i}\beta})$). Thus, we may parametrise the bundle $SL(2,\mathbb{R})$ $(U(1), SL(2,\mathbb{R})/U(1) = \Omega^+)$ by means of the (principal bundle) local chart

$$\Phi_{y_3^+}^{-1} : U_{y_3^+} \times U(1) \to \pi^{-1}(Uy_3^+) \qquad (y_1, y_2, \eta) \to (z_1, z_2) \in SL(2,\mathbb{R})$$

defined by

$$z_1 = \left(\frac{1 + (1 + y_1^2 + y_2^2)^{1/2}}{2}\right)^{1/2} \eta \equiv \left(\frac{1 + (1 + 2z^*z)^{1/2}}{2}\right)^{1/2} \eta \tag{2.5a}$$

$$z_2 = \left(\frac{1 + (1 + y_1^2 + y_2^2)^{1/2}}{2}\right)^{-1/2} \frac{y_2 - \mathrm{i}y_1}{2}\,\eta \equiv \left(\frac{1 + (1 + 2z^*z)^{1/2}}{2}\right)^{-1/2} \frac{z}{\sqrt{2}}\,\eta \tag{2.5b}$$

where $U_{y_3^+}$ is an open set containing $y_3 \in \Omega^+$ and in the expressions above we have introduced

$$z \equiv \bar{\xi}\frac{(\sigma_1 + \mathrm{i}\sigma_2)}{\sqrt{2}}\xi = \frac{y_2 - \mathrm{i}y_1}{\sqrt{2}} \qquad z^* \equiv \bar{\xi}\frac{(-\sigma_1 + \mathrm{i}\sigma_2)}{\sqrt{2}}\xi = \frac{y_2 + \mathrm{i}y_1}{\sqrt{2}}. \tag{2.6}$$

It is obvious that $\Phi^{-1}_{y_3^+}(y_1, y_2; \eta\eta') = \Phi^{-1}_{y_3^+}(y_1, y_2; \eta)\eta'$ and that the identity of the group ($z_1 = 1$, $z_2 = 0$) is mapped by $\Phi_{y_3^+}$ onto $y = (0, 0, 1)$, so that by projecting onto the y_1, y_2 plane we have obtained a local chart at the unity. By extending $U_{y_3^+}$ to the whole Ω^+ the chart becomes a global one. (This is in contrast with the non-trivial structure of the Hopf bundle SU(2, $\mathbb{C}$) (U(1), S^2) which may be derived along similar lines by defining $\bar{\xi} \equiv \xi^\dagger \sigma^0$ and removing the i in (2.3) [18]). The inverse of (2.5) is given by $\Phi_{y_3^+}$: $(z_1, z_2) \to (y_1, y_2, \eta)$ where y_1 and y_2 are given by (2.3):

$$(y_1, y_2, y_3) = (\mathrm{i}(z_1^* z_2 - z_1 z_2^*), z_1 z_2^* + z_1^* z_2, z_1 z_1^* + z_2 z_2^*) \tag{2.7a}$$

and η is obtained from (2.5a):

$$\eta = z_1 \left(\frac{2}{1 + z_1^* z_1 + z_2^* z_2} \right)^{1/2} = \frac{z_1}{|z_1|} \equiv \mathrm{e}^{\mathrm{i}\beta}. \tag{2.7b}$$

The group law in terms of minimal coordinates (y_1, y_2, η) or (z, z^*, η) is now derived† from (2.2) with the result

$$z'' = z'\eta^{-2} + z\varkappa' + \frac{z}{a^2(1+\varkappa)} [z^* z' \eta^{-2} + z'^* z \eta^2]$$

$$z^{*\prime\prime} = z^{*\prime}\eta^2 + z^*\varkappa' + \frac{z^*}{a^2(1+\varkappa)} [z z^{*\prime} \eta^2 + z' z^* \eta^{-2}]$$

$$\eta'' = \left(\frac{2}{1+\varkappa''}\right)^{1/2} \left[\left(\frac{1+\varkappa'}{2}\right)^{1/2} \left(\frac{1+\varkappa}{2}\right)^{1/2} \eta'\eta + \left(\frac{2}{1+\varkappa'}\right)^{1/2} \left(\frac{2}{1+\varkappa}\right)^{1/2} \frac{z' z^*}{2a^2} \eta' \eta^* \right] \tag{2.8a}$$

where in the above expressions $\varkappa$, $\varkappa''$ are merely shorthand for

$$\varkappa \equiv y_3 \equiv \left(1 + \frac{2z^* z}{a^2}\right)^{1/2} \qquad \varkappa'' = \varkappa'\varkappa + \frac{1}{a^2}(z^* z' \eta^{-2} + z'^* z \eta^2) \tag{2.8b}$$

and where we have introduced the parameter $a^2 > 0$ by means of the redefinitions $z \to z/a$, $(y_1, y_2) \to (y_1, y_2)/a$. This will be relevant in the contraction ($a^2 \to \infty$) associated with the non-relativistic limit (section 3). By taking $[a^2] = [\hbar]$, the variables y_1, y_2, z, z^* have now the dimensions of $(\text{action})^{1/2}$ (the interplay between the assignations of physical dimensions and the contraction process is briefly discussed in [13]).

The pseudoextension SL(2, $\mathbb{R}$)$\tilde{\otimes}$U(1) is now defined, with $\tilde{g} = (g, \zeta) \in$ SL(2, $\mathbb{R}$)$\tilde{\otimes}$ U(1) by the group law

$$(g'', \zeta'') \equiv \tilde{g}'' = (g' * g, \zeta'\zeta\sigma_{\text{cob}}(g', g)) \qquad g \in \mathrm{SL}(2, \mathbb{R}), \zeta \in \mathrm{U}(1) \tag{2.9a}$$

where $g'' = g' * g$ is given by (2.7),

$$\sigma_{\text{cob}} = (\eta''\eta'^{-1}\eta^{-1})^{-2a^2/\hbar} \tag{2.9b}$$

is the 2-coboundary generated‡ by $\exp(\mathrm{i}\delta(g)/\hbar) = \eta^{-2a^2/\hbar}$ and η'' in (2.9b) is given

† Because the group action leaves Ω^+ invariant, the group action $y' = g(z_1, z_2)y$ gives a realisation of the two-to-one homomorphism between SL(2, $\mathbb{R}$) and the isometry (the tridimensional 'Lorentz') group of the metric $(+, -, -)$.

‡ The expression of the coboundary generated by $\delta(g)$ is

$$\xi_{\text{cob}}(g', g) = \delta(g' * g) - \delta(g') - \delta(g) \qquad \sigma_{\text{cob}} \equiv \exp[(\mathrm{i}/\hbar)\xi_{\text{cob}}].$$

in (2.8a). The above coboundary has been chosen in such a way that it becomes a non-trivial 2-cocycle in the $a \to \infty$ limit, in which (2.8) *plus* (2.9) will give the group law of a non-trivial extension by U(1).

We remark that the exponent in (2.9b) (the winding number) has to be an integer. Later on (in (4.2b)) we shall relate a with the particle mass and the radius of the anti-de Sitter spacetime. Thus, the definition (2.9b) introduces a quantisation condition for the product mcR, where R is the (real) anti-de Sitter radius.

Let us identify the quantum group associated with the $a \to \infty$ limit of the SL(2, ℝ)$\tilde{\otimes}$ U(1) group. From (2.8) we obtain, by using that $y_3 \to 1$,

$$C'' = C'\eta^{-2} + C \qquad C^{*''} = C^{*'}\eta^2 + C^* \qquad \eta'' = \eta'\eta \tag{2.10a}$$

where we have relabelled the variables after the contraction ($z \to C, z^* \to C^*$). Analogously, (2.9b) requires computing the limit of

$$\left\{\left(\frac{2}{1+\varkappa''}\right)^{1/2}\left[\left(\frac{1+\varkappa'}{2}\right)^{1/2}\left(\frac{1+\varkappa}{2}\right)^{1/2}+\left(\frac{2}{1+\varkappa'}\right)^{1/2}\left(\frac{2}{1+\varkappa}\right)^{1/2}\frac{z^*z'}{2a^2}\eta^{-2}\right]\right\}^{-2a^2/\hbar}$$

which in (2.9a) gives

$$\xi'' = \xi'\xi \exp\left[\mathrm{i}\left(\frac{\mathrm{i}}{2\hbar}(C'C^*\eta^{-2} - CC'^*\eta^2)\right)\right] \tag{2.10b}$$

where the exponential defines the *non-trivial* cocycle. Taken together, (2.10a, b) define the quantum group of the non-relativistic oscillator after identifying η^2 with $e^{i\omega t}$, i.e. $\beta = \frac{1}{2}\omega t$ (2.7b), t being the time. This quantum group was discussed at length in [3-5]. In terms of the variables q, p, the other group parameters C, C^* may be written as

$$C = \left(\frac{m}{2\omega}\right)^{1/2}(\omega x + \mathrm{i}p/m) \qquad C^* = \left(\frac{m}{2\omega}\right)^{1/2}(\omega x - \mathrm{i}p/m). \tag{2.11}$$

To identify the group SL(2, ℝ)$\tilde{\otimes}$U(1) with the symmetry of the relativistic harmonic oscillator another limit can be performed: the $\omega \to 0$ limit. We shall show, by using the configuration space variables, that the contraction limit $\omega \to 0$ describes the free relativistic particle (section 4). Thus, we conclude that we may identify the group (2.8), (2.9) with the quantum group of the relativistic harmonic oscillator.

3. The quantum system associated with SL(2, ℝ)$\tilde{\otimes}$U(1)

We now apply the GAQ. The first essential ingredients of the formalism are the left-invariant vector fields (LIVF), which define restrictions (polarisations) on wavefunctions, and the right ones (RIVF), which define the action of the group on the system in a way compatible with the restrictions (LIVF and RIVF commute). By taking derivatives in the group law ((2.8) and (2.9)), with respect to the unprimed parameters at the identity, we obtain†

$$
\begin{aligned}
\tilde{X}^{\mathrm{L}}_{(z)} &= \varkappa\frac{\partial}{\partial z}+\mathrm{i}\frac{1}{2a^2}\frac{z^*}{1+\varkappa}\left(\mathrm{i}\eta\frac{\partial}{\partial\eta}\right)-\mathrm{i}\frac{z^*}{(1+\varkappa)}\Xi \\
\tilde{X}^{\mathrm{L}}_{(z^*)} &= \varkappa\frac{\partial}{\partial z^*}-\mathrm{i}\frac{1}{2a^2}\frac{z}{1+\varkappa}\left(\mathrm{i}\eta\frac{\partial}{\partial\eta}\right)+\mathrm{i}\frac{z}{(1+\varkappa)}\Xi \\
\tilde{X}^{\mathrm{L}}_{(\eta)} &= \mathrm{i}\eta\frac{\partial}{\partial\eta}-\mathrm{i}2z\frac{\partial}{\partial z}+\mathrm{i}2z^*\frac{\partial}{\partial z^*} \\
\tilde{X}^{\mathrm{L}}_{(\zeta)} &= \frac{\mathrm{i}}{\hbar}\zeta\frac{\partial}{\partial\zeta}\equiv\Xi
\end{aligned}
\tag{3.1a}
$$

where Ξ is the generator of the U(1) factor in the (pseudo)extension. In the same way, taking now derivatives with respect to the primed parameters we obtain the RIVF:

$$
\tilde{X}^{\mathrm{R}}_{(z)} = \eta^{-2}\frac{1}{2(1+\varkappa)}\left[(1+\varkappa)^2\frac{\partial}{\partial z}+\frac{2}{a^2}z^{*2}\frac{\partial}{\partial z^*}-\mathrm{i}\frac{z^*}{a^2}\left(\mathrm{i}\eta\frac{\partial}{\partial\eta}\right)+\mathrm{i}2z^*\Xi\right]
$$

$$
\tilde{X}^{\mathrm{R}}_{(z^*)} = \eta^{2}\frac{1}{2(1+\varkappa)}\left[(1+\varkappa)^2\frac{\partial}{\partial z^*}+\frac{2}{a^2}z^{2}\frac{\partial}{\partial z}+\mathrm{i}\frac{z}{a^2}\left(\mathrm{i}\eta\frac{\partial}{\partial\eta}\right)-\mathrm{i}2z\Xi\right] \tag{3.1b}
$$

$$
\tilde{X}^{\mathrm{R}}_{(\eta)} = \mathrm{i}\eta\frac{\partial}{\partial\eta} \qquad \tilde{X}^{\mathrm{R}}_{(\zeta)} = \frac{\mathrm{i}}{\hbar}\zeta\frac{\partial}{\partial\zeta}.
$$

The LIVF (3.1a) fulfil the following commutation relations:

$$
\begin{aligned}
&[\tilde{X}^{\mathrm{L}}_{(\eta)},\tilde{X}^{\mathrm{L}}_{(z)}] = 2\mathrm{i}\tilde{X}^{\mathrm{L}}_{(z)} \\
&[\tilde{X}^{\mathrm{L}}_{(\eta)},\tilde{X}^{\mathrm{L}}_{(z^*)}] = -2\mathrm{i}\tilde{X}^{\mathrm{L}}_{(z^*)} \qquad [\Xi,\text{all}]=0
\end{aligned}
\tag{3.2a}
$$

$$
[\tilde{X}^{\mathrm{L}}_{(z)},\tilde{X}^{\mathrm{L}}_{(z^*)}] = \frac{-\mathrm{i}}{2a^2}\tilde{X}^{\mathrm{L}}_{(\eta)}+\mathrm{i}\Xi \tag{3.2b}
$$

while the RIVF satisfy the same relations (3.2) but for an additional global minus sign on the RHS; of course [any $\tilde{X}^{\mathrm{L}}$, any $\tilde{X}^{\mathrm{R}}$] = 0. If we ignore the Ξ term in the commutator $[\tilde{X}_{(z)},\tilde{X}_{(z^*)}]$, which carries the Planck constant, the commutators of (3.2) are those of the sl(2, $\mathbb{R}$) algebra in the standard basis. In the limit $a\to\infty$, (3.2) gives

$$
\begin{aligned}
\tilde{X}^{\mathrm{L}}_{(C)} &= \frac{\partial}{\partial C}-\frac{\mathrm{i}}{2}C^+\Xi \\
\tilde{X}^{\mathrm{L}}_{(C^+)} &= \frac{\partial}{\partial C^+}+\frac{\mathrm{i}}{2}C\Xi \qquad \tilde{X}^{\mathrm{L}}_{(\zeta)}=\Xi \\
\tilde{X}^{\mathrm{L}}_{(\eta)} &= \mathrm{i}\eta\frac{\partial}{\partial\eta}-2\mathrm{i}C\frac{\partial}{\partial C}+2\mathrm{i}C^+\frac{\partial}{\partial C^+}
\end{aligned}
\tag{3.3}
$$

† Although the Ξ term of the extension may be computed directly from (2.9), it is faster to use that

$$
\tilde{X}^{\mathrm{L}}_{(g')} = X^{\mathrm{L}}_{(g')}+\left[L_{X_{g'}}\delta(g)-\left[\frac{\partial\delta(g)}{\partial g'}\right]_{g'=e}\right]\Xi
$$

[20], where $X_{g'}$ is the vector field of the unextended sl(2, $\mathbb{R}$) algebra associated with the group parameter g' and L is the Lie derivative; in this way, one avoids using the expression of the coboundary (2.9b). Note that, although we use the global parameter η, the local parameter is the angle β; we have $\partial/\partial\beta=\mathrm{i}\eta\partial/\partial\eta$ (the same comment applies to the other compact parameter ζ). The exponent $\delta(g)$ is in our case $\delta(g)=-2a^2\beta=-a^2\omega t$.

$$\left(\tilde{X}^{\mathrm{L}}_{(t)}=\frac{\partial}{\partial t}-\mathrm{i}\omega C\frac{\partial}{\partial C}+\mathrm{i}\omega C^{+}\frac{\partial}{\partial C^{+}}\right)$$

which generate the Lie algebra of (2.10*a*, *b*) which is given by (3.2*a*) and

$$[\tilde{X}^{\mathrm{L}}_{(C)}, \tilde{X}^{\mathrm{L}}_{(C^{+})}]=\mathrm{i}\Xi \tag{3.4}$$

which replaces (3.2*b*).

The canonical left-invariant 1-form on SL(2, ℝ)$\tilde{\otimes}$U(1) is a sl(2, ℝ)$\tilde{\otimes}$u(1)-valued 1-form (see, e.g., [22]). Among its four components, we are interested here in the (vertical) component defined by being the 1-form dual to the vector field of the U(1) generator. As can be shown from the general theory such a form provides the quantisation form, as its expression will make it evident. This is determined by $\Theta(\Xi)=1$, $\Theta(\tilde{X}^{\mathrm{L}}_{(z,z^*,\eta)})=0$, and is given by

$$\Theta=\frac{1}{(1+\varkappa)}\mathrm{i}[z^*\mathrm{d}z-z\mathrm{d}z^*]-2[\varkappa-1]a^2\frac{\mathrm{d}\eta}{\mathrm{i}\eta}+\hbar\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}. \tag{3.5}$$

In the limit $a\to\infty$ in which $\varkappa\sim 1+zz^*/a^2$, Θ becomes

$$\Theta_{\mathrm{NR}}=\tfrac{1}{2}\mathrm{i}[C^{+}\mathrm{d}C-C\mathrm{d}C^{+}]-2C^{+}C\mathrm{d}\beta+\hbar\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{3.6}$$

which is the quantisation form ($2\mathrm{d}\beta=\omega\mathrm{d}t$) of the non-relativistic harmonic oscillator as was obtained [3] from the vertical part of the canonical left-invariant 1-form for the group (2.10). It is interesting to remark, however, that the 1-forms (3.5) and (3.6), which are unambiguously given by the group quantisation formalism from their respective groups, are related by a simple change of variables

$$C=\left(\frac{2}{1+y_3}\right)^{1/2}z \qquad C^{+}=\left(\frac{2}{1+y_3}\right)^{1/2}z^* \tag{3.7}$$

where again y_3 is given by (2.8*b*) and which identifies C and z in the $a^2\to\infty$ limit. This indeterminancy in the association of (quantisation form)→(quantum system), which appears in the conventional geometric quantisation formalism where (3.5) and (3.6) are the *only* starting points, is solved when it is replaced by the correspondence *quantum group→quantum system*, and the rest of our GAQ (in particular, the group definition of the polarisation subalgebra) is taken into account (see below and section 4).

We now proceed to determine the wavefunction of the quantum system associated with the quantum group SL(2, ℝ)$\tilde{\otimes}$U(1). For it, we have to determine the characteristic module and the maximal polarisation subalgebra. The characteristic module C_Θ of (3.5) polarisation subalgebra is defined as the module generated by the vector fields X which satisfy $i_X\Theta=0$, $i_X\mathrm{d}\Theta=0$. From (3.5) and

$$\mathrm{d}\Theta=\frac{\mathrm{i}}{y_3}\left[\mathrm{d}z^*\wedge\mathrm{d}z+2\mathrm{i}(z\mathrm{d}z^*+z^*\mathrm{d}z)\wedge\frac{\mathrm{d}\eta}{\mathrm{i}\eta}\right] \tag{3.8}$$

it is found that the characteristic module is generated by the LIVF associated with the compact (periodic) evolution parameter β:

$$C_\Theta=\left\langle\tilde{X}^{\mathrm{L}}_{(\eta)}=\mathrm{i}\eta\frac{\partial}{\partial\eta}-2\mathrm{i}z\frac{\partial}{\partial z}+2\mathrm{i}z^*\frac{\partial}{\partial z^*}\right\rangle. \tag{3.9}$$

The quantum system is now characterised by wavefunctions $\Psi(z, z^*, \eta, \zeta)$ on the group manifold $SL(2,\mathbb{R})\tilde{\otimes}U(1)$ which satisfy $\Xi\Psi = i\Psi$, i.e. $\Psi = \zeta\Psi(z, z^*, \eta)$, and

$$\tilde{X}^{L}_{(\eta)}\psi = 0 \qquad \tilde{X}\psi = 0 \qquad \forall\tilde{X} \in \text{polarisation sublagebra.} \tag{3.10}$$

The polarisation subalgebra, which is defined [3-5] as the maximal horizontal left-subalgebra containing $\tilde{X}^{L}_{\eta}$, is obtained by adding $\tilde{X}^{L}_{(z)}$ (see (3.2a)). It is at this point, when the group theoretical definition of polarisation is introduced, where the ambiguity implied by the change of variables (3.7) and which is inherent to the geometric quantisation form [see, e.g., 23] is solved. Although (3.7) brings the evolution vector field $\tilde{X}^{L}_{(\eta)}$ of (3.9) ((3.1a)) into the $\tilde{X}^{L}_{(\eta)}$ (3.3) of the non-relativistic case as it did with the quantisation forms, this is not the case with the other vector field of the polarisation subalgebra, $\tilde{X}^{L}_{(z)}$ (3.31a). This has to be so, because a redefinition of the group parameters by means of a change of variables cannot alter the group structure (explicitly, it is seen that such a change, which does not involve η, cannot eliminate the term in $i\eta\partial/\partial\eta$ of $\tilde{X}^{L}_{(z)}$ (3.1a) to get $\tilde{X}^{L}_{(c)}$ (3.3)). The wavefunctions of the quantum system are thus U(1)-equivariant functions ($\Xi\Psi = i\Psi$) which satisfy $\tilde{X}^{L}_{(\eta)}\Psi = 0$ *and* $\tilde{X}^{L}_{(z)}\Psi = 0$. Writing

$$\psi = \zeta(1+\varkappa)^{-\gamma a^2/\hbar}\varphi(z, z^*, \beta) \tag{3.11a}$$

$\tilde{X}^{L}_{(z)}\Psi = 0$ gives $\gamma = 1$ and

$$\varkappa\frac{\partial\varphi}{\partial z} + \frac{i}{2a^2}\frac{z^*}{1+\varkappa}\frac{\partial\varphi}{\partial\beta} = 0. \tag{3.11b}$$

$\tilde{X}^{L}_{(\eta)}\Psi = 0$ then gives

$$\frac{\partial\varphi}{\partial\beta} - 2iz\frac{\partial\varphi}{\partial z} + 2iz^*\frac{\partial\varphi}{\partial z^*} = 0. \tag{3.11c}$$

The final solution is given by a superposition of solutions of the type (see [21] for the SU(2) case)

$$\psi_n = 2^{a^2/\hbar}(1+\varkappa)^{-a^2/\hbar}[\exp(-2i\beta)(1+\varkappa)^{-1}z^*]^n. \tag{3.12}$$

In the limit $a\to\infty$ (3.12) gives

$$\Phi_n = [\exp(-C^+C/\hbar)]\exp(-i\omega t)C^{+n}. \tag{3.13}$$

We notice in (3.13) the appearance of the Bargmann weight factor for the non-relativistic harmonic oscillator. Thus, the wavefunctions (3.12), as well as (3.11b, c), go to their non-relativistic counterparts [3-5].

In the GAQ, the basic quantum operators are given, apart from a constant factor involving $\hbar$, by the RIVF. The one which generates the compact ($U(1)\subset SL(2,\mathbb{R})$) time translations is given by $\tilde{X}^{R}_{(\eta)}$ (3.1b); in terms of $t = 2\beta/\omega$, $\hat{E} = i\hbar\omega/2\tilde{X}^{R}_{(\eta)} = i\hbar\partial/\partial t$† and we obtain $\hat{E}\Psi_n = n\hbar\omega\Psi_n$. We note at this stage that the non-appearance of the vacuum energy additive term, $\hbar/2$, is a consequence of the fact that the GAQ provides the correct ('normal-ordered') form for the quantum operators. The action of the rest of the operators is $\tilde{X}^{R}_{(z)}\Psi_n \sim (n-1)\Psi_{n+1}$ and $\tilde{X}^{R}_{(z^*)}\Psi_n \sim n\Psi_{n-1}$. For the non-relativistic oscillator, for instance, the operators $\hat{C}$ and $\hat{C}^+$ are given by [4, 5]

$$\hat{C}^+ = -\hbar\exp(-i\omega t)\left[\frac{\partial}{\partial C} + \tfrac{1}{2}iC^+\Xi\right] \qquad \hat{C} = \hbar\exp(i\omega t)\left[\frac{\partial}{\partial C^+} - \tfrac{1}{2}iC\Xi\right]. \tag{3.14}$$

† The *factor* $\frac{1}{2}$ in the definition of E has been introduced because the initial group law (2.8a) had to be given in terms of η^2 (for the z'' and z''^* part) to avoid a square root $\eta^{1/2}$ in the η'' part.

Using (3.14) we find

$$\hat{C}^+\Phi_n = \hbar\Phi_{n+1} \qquad \hat{C}\Phi_n = n\hbar\Phi_{n-1} \qquad \hat{C}^+\hat{C}\Phi_n = n\hbar\Phi_n. \tag{3.15}$$

Thus, we conclude that our energy operator is just given by $\hat{C}^+\hat{C}$ and differs from the prescription $\frac{1}{2}(\hat{C}^+\hat{C}+\hat{C}\hat{C}^+)$ in the term $(\hbar\omega)/2$. The situation is similar in the relativistic case (3.1*b*) and (3.12) provided we compare with the squared energy operator†.

4. The relativistic harmonic oscillator in configuration space

Let us make the change of variables

$$\begin{gathered}(z, z^*, \beta)\rightarrow(x, p, x^0\equiv c\tilde{t})\\ z=\left(\frac{m}{2\omega}\right)^{1/2}(\omega x+\mathrm{i}p/m) \qquad z^*=\left(\frac{m}{2\omega}\right)^{1/2}(\omega x-\mathrm{i}p/m)\end{gathered} \tag{4.1a}$$

$$2\beta=\sin^{-1}[(Px-\lambda p)(p^2/m\omega+m\omega x^2)^{-1}] \tag{4.1b}$$

where $\omega = mc^2/a^2$ and P and λ are defined in (4.2*d*) below. The change (4.1) transforms the group law (2.8*a*) into

$$\begin{aligned}p''&=\frac{1}{mc}(p'p^0+pP'^0)-\frac{K}{\Lambda}(p^0x^0-px)x'\\ x''^0&=\frac{p'^0x^0}{mc}+\frac{xP'}{mc}+\Lambda x'^0\\ x''&=\frac{p'x^0}{mc}+\frac{xP'^0}{mc}+\Lambda x'\end{aligned} \tag{4.2a}$$

where

$$K\equiv\frac{1}{R^2}\equiv\frac{\omega^2}{c^2} \qquad \Lambda\equiv[1-K((x^0)^2-x^2)]^{1/2} \tag{4.2b}$$

so that R has dimensions of length (de Sitter radius), p^0 is the solution of the mass shell condition (see (4.9*a*)),

$$p^0=\frac{Kpxx^0+\Lambda P^0}{1+Kx^2} \qquad \left((p^0)^2-p^2+\frac{K}{\Lambda^2}(p^0x^0-px)^2=m^2c^2\right) \tag{4.2c}$$

and

$$\begin{aligned}P^0&\equiv\frac{p^0+Kx(p^0x-px^0)}{\Lambda}=(m^2c^2+p^2+m^2\omega^2x^2)^{1/2}\\ P&\equiv\frac{p+Kx^0(p^0x-px^0)}{\Lambda} \qquad \lambda\equiv(p^0x-px^0)/mc.\end{aligned} \tag{4.2d}$$

(The definitions (4.2*d*) are introduced because we shall see (4.11) that P^0, P and λ are Noether constants of the motion.) The quantisation condition associated with the

† To be more precise, we must compare the expression $\frac{1}{2}(\hat{z}^+\hat{z}+\hat{z}\hat{z}^+)$ with the squared energy operator $\hat{E}^2=(\tilde{X}^{\mathrm{R}}_{(\eta)}-\Xi)^2$. We have now $\hat{E}^2=\frac{1}{2}(\hat{z}^+\hat{z}+\hat{z}\hat{z}^+)+(\frac{1}{2}\hbar\omega)^2$. The change $\tilde{X}^{\mathrm{R}}_{(\eta)}\rightarrow\tilde{X}^{\mathrm{R}}_{(\eta)}-\Xi$ destroys the pseudocohomology and puts the observables in a standard form.

winding number in (2.9*b*) tells us, using (4.2*b*), that mcR/h has to be a half-integer. This number relates the mass mc with the mass correction which appears in the *anti*-de Sitter Dirac equation in a $(1+2)$ flat space [24].

The SL(2, ℝ) group law (4.2) has now to be completed with the 2-coboundary (2.9) λ defining the U(1)-pseudoextension in the new coordinates (p, x^0, x, ζ). In them, the LIVF and RIVF of $SL(2,\mathbb{R})\tilde{\otimes}U(1)$ and their commutation relations are given by

$$\begin{aligned} \tilde{X}^{\mathrm{L}}_{(p)} &= \frac{P^0}{mc}\frac{\partial}{\partial p}+\frac{mcx}{P^0+mc}\Xi \\ \tilde{X}^{\mathrm{L}}_{(x^0)} &= -mcKx\frac{\partial}{\partial p}+\frac{p^0}{mc}\frac{\partial}{\partial x^0}+\frac{p}{mc}\frac{\partial}{\partial x} \\ \tilde{X}^{\mathrm{L}}_{(x)} &= \frac{P}{mc}\frac{\partial}{\partial x^0}+\frac{P^0}{mc}\frac{\partial}{\partial x}-\frac{mcp}{mc+P^0}\Xi \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tilde{X}^{\mathrm{R}}_{(p)} &= \frac{p^0}{mc}\frac{\partial}{\partial p}+\frac{x}{mc}\frac{\partial}{\partial x^0}+\frac{x^0}{mc}\frac{\partial}{\partial x}-\frac{\lambda mc}{P^0+mc}\Xi \\ \tilde{X}^{\mathrm{R}}_{(x^0)} &= \Lambda\frac{\partial}{\partial x^0} \\ \tilde{X}^{\mathrm{R}}_{(x)} &= -\frac{K}{\Lambda}(p^0x^0-xp)\frac{\partial}{\partial p}+\Lambda\frac{\partial}{\partial x}+\frac{Pmc}{P^0+mc}\Xi \end{aligned} \tag{4.4}$$

$$\begin{aligned} &[\tilde{X}^{\mathrm{L}}_{(p)}, \tilde{X}^{\mathrm{L}}_{(x)}] = \frac{1}{mc}\tilde{X}^{\mathrm{L}}_{(x^0)}+\Xi \qquad && [\tilde{X}^{\mathrm{L}}_{(p)}, \tilde{X}^{\mathrm{L}}_{(x^0)}] = \frac{1}{mc}\tilde{X}^{\mathrm{L}}_{(x)} \\ &[\tilde{X}^{\mathrm{L}}_{(x^0)}, \tilde{X}^{\mathrm{L}}_{(x)}] = mcK\tilde{X}^{\mathrm{L}}_{(p)} && [\Xi, \mathrm{all}] = 0. \end{aligned} \tag{4.5}$$

The quantisation form is found to be

$$\Theta = -\frac{1}{\Lambda}(P^0\,\mathrm{d}x^0 - P\,\mathrm{d}x)+\frac{2mc^2}{\omega}\mathrm{d}\beta+\hbar\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{4.6}$$

where, for the sake of simplicity, we have retained the old time variable β in the exact $\mathrm{d}\beta$ term.

We now show again that the group structure provides all the information which is required for the description of the dynamical system in configuration space. In particular, it *defines* spacetime itself and its metric. To see it, we now look for the trajectories of the LIVF generating the characteristic module of (4.6), $X^{\mathrm{L}}_{(x^0)}$. From its expression in (4.3) we find

$$\frac{\mathrm{d}x^0}{\mathrm{d}\tau}=\frac{p^0}{m} \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau}=\frac{p}{m} \qquad \frac{\mathrm{d}p}{\mathrm{d}\tau}=-mc^2Kx. \tag{4.7}$$

From (4.7) and (4.2*c*) it follows that $\mathrm{d}p^0/\mathrm{d}\tau=-Kmcx^0$ and

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}=-Kx^0 \qquad \frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}=-Kx. \tag{4.8}$$

We thus obtain the equations of the geodesics of a particle in a bidimensional anti-de

Sitter universe embedded in a flat three-dimensional space of coordinates (x^0, x, w), $K\eta_{\mu\nu}x^\mu x^\nu + w^2 = 1$, characterised by the metric (see, e.g., [17])

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{K}{(1 - K\eta_{\rho\sigma}x^\rho x^\sigma)}\,\eta_{\mu\lambda}x^\lambda\eta_{\nu\chi}x^\chi \tag{4.9a}$$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - Kx^\mu x^\nu \qquad (\mu, \nu = 0, 1) \tag{4.9b}$$

where $\eta_{\mu\nu} = (+, -)$. Indeed, rewriting (4.6) in the usual form $\Theta = -p_\mu\, dx^\mu + \hbar d\zeta/i\zeta$ by ignoring here the inessential exact part in $d\beta$ we obtain

$$p_\mu \equiv \frac{\eta_{\mu\nu}}{\Lambda}P^\nu \equiv g_{\mu\nu}p^\nu \tag{4.10}$$

where $g_{\mu\nu}$ is given by (4.9a). The mass shell condition (4.2c) is then $g_{\mu\nu}p^\mu p^\nu = m^2c^2$.

We shall not discuss the classical limit. Ignoring the U(1) part $\hbar d\zeta/i\zeta$ in (4.6) might appear as a simple way of performing the $\hbar \to 0$ limit; nevertheless, in the GAQ the classical limit is more properly obtained [3-5] by substituting the additive group $\mathbb{R}$ for U(1) (see the appendix). Similarly, we shall not discuss the quantisation in this (evolution space) parametrisation. Indeed, in this case Kähler-like polarisations are required (SL(2, $\mathbb{R}$)/U(1) is a Kähler manifold) and this would lead us back to repeat the quantisation already performed in Section 3. Let us compute instead the Noether invariants, which in the GAQ are obtained [5, 25, appendix] by computing $i_{\tilde{X}^R}\Theta$ from (4.4) and (4.6). We obtain†

$$\begin{aligned} i_{\tilde{X}^R_{(p)}}\Theta &= (p^0x - x^0p)/mc \equiv \lambda \\ i_{\tilde{X}^R_{(x)}}\Theta &= (P^0 - mc) \\ i_{\tilde{X}^R_{(x)}}\Theta &= -P \end{aligned} \tag{4.11}$$

which allow us to rewrite the mass-shell condition (4.2c) in terms of constants of the motion:

$$(P^0)^2 - P^2 - Km^2c^2\lambda^2 = m^2c^2. \tag{4.12}$$

We may use the Noether invariants (4.11) to give the generalised Poisson brackets of our formalism. (They are generalised because, on the group manifold, $d\Theta$ is not a symplectic form.) They are given by

$$\{i_{\tilde{X}^R_{(a)}}\Theta, i_{\tilde{X}^R_{(b)}}\Theta\} \equiv i_{[\tilde{X}^R_{(a)}, \tilde{X}^R_{(b)}]}\Theta \tag{4.13}$$

and reproduce the commutation relations of the group algebra. By taking the quotient by the characteristic module $\tilde{X}^L_{(x^0)}$ (something the GAQ does *not* require for the quantisation) the ordinary symplectic Poisson brackets may be recovered [3-5].

To conclude, we mention the $c \to \infty$ and $\omega \to 0$ limits in the present parametrisation. It is simple to check that the $c \to 0$ limit reduces the last line of (4.1b) to $2\beta = \sin^{-1}\omega\tilde{t}$ and, since $2\beta = \omega t$, the ordinary time is given by $t = (\sin^{-1}\omega\tilde{t})/\omega$. Then (4.2$a$), (4.3)-(4.6) and (4.11) go to their analogues for the non-relativistic harmonic oscillator [3-5],

† It may be easily checked that the three constants of the motion λ, P^0, P below coincide with those given in [24], formulae (4.8), (4.9) ($\xi_5 \to wc/\omega$, $\xi^0 = x^0$, $\xi^1 = x$).

a dynamical system which may also be interpreted as a free non-relativistic anti-de Sitter particle. (For this limit, the exact $d\beta$ part in (4.6) cannot be ignored, because this part comes from the coboundary generating a cocycle when $c \to \infty$ and thus contributes to the limit of Θ in an essential manner.) The limit $\omega \to 0$ ($R \to \infty$; $K \to 0$; $\Lambda \to 1$), on the other hand, makes the metric (4.9) flat and transforms (4.2*a*), (4.3)-(4.6), (4.11) and (4.12) to those of a free relativistic particle [13] whose energy has the rest mass substracted, as is evident from (4.2*d*), (4.6) and the second equation in (4.11).

5. Conclusions and outlook

We have completely determined the dynamics associated with the SL(2, $\mathbb{R}$) group. Two different physical aspects have been considered. First, and by means of the $\omega^2 \to 0$ and $c \to \infty$ group contractions (performed on both the group law and the wavefunctions), we have seen how the SL(2, $\mathbb{R}$) dynamics corresponds to what we may call a relativistic harmonic oscillator. The related Fock space in its present form is of a great interest as it is the starting point for the study of the representation of the Virasoro group (see [26] and references therein). On the other hand, the same group has been seen to describe the quantum dynamics of a 'free' particle moving in $(1+1)$ anti-de Sitter spacetime. In particular, we have obtained the metric and the geodesic equations from just the group manifold. The analysis of these equations, together with the Noether invariants, also defined in a natural way, permits the definition of a configuration space for the above constructed harmonic oscillator. The Noether invariant P^o associated with the time translations has a simple expression in terms of the 'potential' $m^2\omega^2x^2$ (4.2*d*) in contrast with the p^o appearing in the mass shell equation (4.2*c*).

The above study of the SL(2, $\mathbb{R}$) group could be extended to more realistic spacetime dimensions by taking the SO(4.1) or/and SO(3.2) groups as the starting symmetry. In so doing, the only problem we would find would be the complexity of the exact group law when written in an adequate parametrisation, i.e. in terms of parameters closely related to physical quantities. Otherwise the quantisation would be achieved by exactly the same steps followed in the previous sections. Furthermore, we could develop the study sketched in [14] concerning all possible dynamics associated with the conformal group SO(4,2). We should then obtain the off-shell dynamics of a free particle moving in de Sitter, anti-de Sitter and Minkowski spacetime, respectively, according to what one-dimensional subgroup of SL(2, $\mathbb{R}$) $\subset$ SO(4,2) is chosen as the structure group of SO(4,2). The group laws, LIVF, RIVF, etc, of the different possible kinematical groups, including the conformal group, have already been derived by using a symbolic computer program [27].

An alternative way of dealing with very involved group laws is also being applied successfully. A formal power series (in the sense of formal groups [28]) can replace the exact group law so that the essential of the dynamics becomes manifest in the few lowest orders. This technique, already used in infinite-dimensional Lie groups [26, 29, 30], may equally be applied to finite-dimensional ones and, in particular, to the largest kinematical group SO(4,2).

The conformal group law will allow us to investigate a different direction also concerning gravitation. The idea is to substitute the SO(4,2) group for the Poincaré group in a field theory of massless particles. For instance, an infinite-dimensional group law has been found [31] which reproduces the (Gupta–Bleuler) quantisation of the (free) electromagnetic field in ordinary Minkowski spacetime. If we replace the Poincaré group contained in this group with the conformal group, the specific conformal transformations could be related to accelerations violating, in a definite way, the classification into creation and annihilation operators defining the Fock space of states. This approach could be an appropriate framework for studying field quantisation in the presence of gravity, Unruh effect [32], etc.

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Appendix. The quantisation form, the classical limit and the Poincaré–Cartan form

In this appendix we explain the connection between the Poincaré-Cartan-Hilbert form of a classical system (see, e.g., [25, 33]) and the canonical form provided by the GAQ, and discuss the derivation of the Noether invariants [5, 25]. Restricting ourselves here to dynamical systems associated with groups which are central entensions or pseudo-extensions (our present case), the GAQ associates the quantum system to the U(1) extension, $\tilde{G}$, and the classical counterpart to the extension by $\mathbb{R}$, $\bar{G}$. In this second case, the equivariance condition $\Xi\Psi = i\Psi$ for the wavefunction is replaced by $\partial S/\partial\chi = 1$ where χ parametrises $\mathbb{R}$ and S is the Hamilton-Jacobi function. Since χ now is not an exponent, it may take the dimensions of an action, and thus $\hbar$ is no longer necessary. The expression of the quantisation form and its classical counterpart may be written as [3-5]

$$\Theta = \Theta_{PC} + \hbar\frac{d\zeta}{i\zeta} \qquad \bar{\Theta} = \Theta_{PC} + d\chi. \tag{A1}$$

Their common term, Θ_{PC}, is the Poincaré-Cartan form. Although the difference between $\bar{\Theta}$ and Θ_{PC} is an exact 1-form, and thus the classical trajectories (in q and p) are the same, the extra term $d\chi$ allows us to define symmetries by the strict invariance condition $L_{\bar{X}}\bar{\Theta} = 0$, where $\bar{X}$ is a RIVF on $\bar{G}$, rather than by $L_X\Theta_{PC} = df$ where X is a RIVF on the unextended group G. In the first case, the Noether invariant is $i_{\bar{X}}\bar{\Theta} = i_X\Theta_{PC} + \bar{X}^{\chi}$; in the second case it has to be defined by $i_X\Theta_{PC} - f$. The presence of both f and the component $\bar{X}^{\chi}(=-f)$ is a consequence of the cohomology (or pseudocohomology [20]) of G, and not of the spacetime itself; this is one of the reasons for trying to build the dynamics on groups and not just on spacetime. Notice, finally, that the Noether invariants are not necessarily associated with $\bar{\Theta}$; obviously, the same result is obtained from $i_{\tilde{X}}\tilde{\Theta}$, as used in section 4.

We refer to [5, 25] for a detailed discussion.

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